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the perfectly formed languages, but the dialects of savage tribes, so with the teacher. In like manner more insight into the philosophy of teaching and of the nature of the mind can be acquired by teaching a class of children to read than in any other grade of work.

BOOLE'S LOGICAL METHOD.

BY GEORGE BRUCE HALSTEAD.

Perhaps the possession of absolute originality cannot be better demonstrated than by breaking through the barriers inside which men have hitherto worked, pushing boldly into what was supposed to be outer void and darkness, and, without hint, without help, opening broad roads and showing fertile fields for wholly new, unsuspected sciences. This did George Boole in more than one direction.

The vast Invariantive Algebra, which is now the foundation rock of modern advance in mathematics, was started by him. Says Salmon (3d ed., p. 103): "What I have called Modern Algebra may be said to have taken its origin from a paper in the Cambridge *Mathematical Journal* for November, 1841, where Dr. Boole established the principles just stated, and made some important applications of them."

Of the same epoch-making character were his extensive contributions to the Calculus of Operations. Again, in 1862, Russell said before the British Association, in regard to the Calculus of Symbols: "It received a fresh impulse from the very remarkable memoir of Prof. Boole (on a 'General Method in Analysis.' Phil. Trans., 1844), in which an algebra of non-commutative symbols was invented and applied." He found many willing and able to follow on these roads, and to settle in the new lands thus laid open; but when, in 1847, he struck the key-note of a generalization of logic, which exhibits it as almost a new science, he seems to have advanced too far beyond his time, and so was left to carry it on alone, which he did in his great work, "The Laws of Thought," published in 1854. That this, at the present moment, instead of being a thing of the past, is just beginning to attract that attention so well deserved by its extraordinary originality and suggestiveness, carries a plain inference in regard to the character of the mind capable of producing it, unaided, a quarter of a century ago.

What, then, was his generalization, and what the method he proposed for the solution of the general problem?[1]

The problem may be very compactly stated, but we cannot guarantee that the reader will be able at once to appreciate its full significance. It is: "Given any assertions, to determine precisely what they affirm, precisely what they deny, and precisely what they leave in doubt, separately and jointly." Or, as Boole himself puts the "statement of the final problem of practical logic. Given a set of premises expressing relations among certain elements, whether things or propositions; required explicitly the whole relation consequent among *any* of those elements, under any proposed conditions and in any proposed form."

That this is vastly more general than anything ever attempted by the old logic, needs no pointing out. Its startling breadth makes it seem, at first, absolutely insoluble. To illustrate this, suppose Boole had, as many cursory readers have supposed, made logic depend on the solution of ordinary algebraic equations. With the world of mathematicians to aid him, he could never have solved his problem; for from its very essence it can make no restrictions as to the number or degrees of equations, and mathematicians have never been able to find a general solution for even the equation of the fifth degree, while some of their greatest have given demonstrations of the impossibility of such solution.

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[1] The *Revue Philosophique* for September 1877, contains an article thirty-three pages long on "La Logique Algèbrique de Boole," by Louis Liard. It is, for the most part, simply a translation of so much of the original, blunders included, into French. Number IV of *Mind*, October, 1876, contained an article of twelve pages on "Boole's Logical System," by J. Venn. This we enthusiastically recommend to our readers. We only wish it had been three times as long, and that the author had entered somewhat more into detail.

In going on to state how Boole actually did accomplish his purpose, we are met at the outset by a difficulty in the shape of a familiar word, which, as used by him, has been by prominent logicians disastrously misconceived. His critics have always used the term "mathematics" as dealing essentially with quantitative specification, and have drawn their arguments from the supposition that Boole was using the term in that sense. Even his friends have made their fight on this assumed line; which accounts for R. Harley's saying "Logic is never identified or confounded with mathematics," and for Mr. Venn's saying "The prevalent notion about Boole probably is that he regarded logic as a branch of mathematics. This is a very natural mistake."

Boole himself says, p. 11: "Whence is it that the ultimate laws of logic are mathematical in their form;" and, p. 12, says again of logic: "But it is equally certain that its ultimate forms and processes are mathematical." The key to the difficulty is contained in one short sentence, which should have been printed in capitals: "It is not of the essence of mathematics to be conversant with the ideas of number and quantity."

This simply means that Boole felt strongly the need of some word broad enough to cover the range of sciences expressible by algebras, and thought the facts justified his taking the old word "mathematics" for such a signification.

He says, in regard to it: "The predominant idea has been that of magnitude, or, more strictly, of numerical ratio." * * * "This conclusion is by no means necessary. We might justly assign it as the definitive character of a true calculus; that it is a method resting upon the employment of symbols, whose laws of combination are known and general, and whose results admit of a consistent interpretation." In this sense he chooses to use the word "mathematical," and in this sense his symbolic logic is as much a branch of mathematics as the ordinary algebra of number.

His broadened use of the word has been accepted by some as meeting a real want, among whom we may mention Professor Benjamin Pierce, who adds: "Qualitative relations can be considered by themselves, without regard to quantity. The algebra of such enquiries may be called logical algebra, of which a fine example is given by Boole." By bearing in mind this point we may avoid this pit, which seems to have rendered dangerous all approach to the work under consideration, and into which Stanley Jevons was one of the first to fall.

In any algebra the laws of combination of symbols are all-important. Upon these depend its particular character and the validity of its processes. So here, in seeking to discover the most natural algebra for logic, though we may convene to represent by letters, x, y, a, b, etc., all ordinary logical classes, we must determine how they combine formally, by careful consideration of the intellectual operations implied in the best use of language as an instrument of reasoning. All thought postulates: I. The law of Identity: $x=x$. II. The law of Contradiction: It is impossible for any being to possess a quality and at the same time not to possess it. III. The law of Excluded Middle: Everything is either x or not x. Reasoning on classes postulates also the axiom: IV. Whatever is predicated of a class may be predicated of the members of that class. Had Boole only referred to these openly, instead of making use of them unconsciously, he would have saved himself a vast amount of trouble and some positive error.

Convening, then, to represent any class by a letter—as, men by a and good things by b—we see that, when these are combined in thought or language, one acts as a selective adjective, and that, whichever this be, the result is the same; so that ba, or "good men," gives us the same collection of individuals as ab, or "human good beings." Using the sign $=$ as meaning, in the most general way, identity, co-existence, or equality, we say $ab=ba$. "We are permitted, therefore, to employ the symbols x, y, a, b, etc., in the place of substantives, adjectives, and descriptive phrases, subject to the rule of interpretation that any expression in which several of these symbols are written together shall represent all the objects or individuals to which their several meanings are together applicable, and to the law that the order in which the symbols succeed each other is indifferent."

Again, to form the aggregate conception of a group of objects consisting of partial groups, we use the conjunctions "and," "or." Convening that the classes so joined are quite distinct, so that no individual is added to himself, we see that these conjunctions hold precisely the same position *formally* as the sign $+$ in the ordinary algebra of number, and, therefore, are represented by that sign. As the order of addition is indifferent, we have $x+y=y+x$; and, from IV, $z(x+y)=zx+zy$. Again, to separate a part from a whole, we express in common language by the sign "except" ($-$), as, "All men except Asiatics." This is our minus. As it is indifferent whether we express excepted cases first or last, we have $x-y=-y+x$, and, from IV, $z(x-y)=zx-zy$.

So we may at once affirm for our logical algebra the validity of the three general axioms:

1. Equals added to equals give equals.
2. Equals multiplied by equals give equals.
3. Equals taken from equals give equals.

Though each of these may be demonstrated for the algebra of logic entirely independently of even the existence of any such thing as the algebra of number, yet we see it actually turns out that, so far, the two algebras are *formally* identical. This may lead the reader to wish that this formal identity had held throughout, so that he might have interpreted his quantitative mathematics directly as so much logic, just as the *same process* may, under one scheme of interpretation, represent the solution of a question on the properties of numbers, under another, that of a geometrical problem, and under a third, that of a problem in dynamics or optics. But let me repeat that, if no different operative law had manifested itself, the algebra of logic, like that of number, would have been stopped short at the equation of the fifth degree, and so its general problem could never have been solved.

Just as the algebra of quaternions differs in one fundamental law from the algebra of number, namely, in its multiplication being non-commutative, so that *ab* does not equal *ba*, so our algebra of logic differs in a law equally fundamental, and from this difference comes the power that, in it, every equation can be solved and every solution interpreted.

The real nature and unavoidable character of this law in our new algebra depend upon the general postulates of thought which we have given; but, unfortunately, Boole, groping in the darkness of a dawning subject, introduced the matter upside down, and so was led into a curious error. He commences thus, p. 31: "As the combination of two literal symbols in the form xy expresses the whole of that class of objects to which the names or qualities represented by x and y are together applicable, it follows that, if the two symbols have exactly the same signification, their combination expresses no more than either of the symbols taken alone would do. That is, $xx=x^2=x$. The law which this expresses is practically exemplified in language. To say 'good good,' in relation to any subject, though a cumbrous and useless pleonasm, is the same as to say 'good.' Thus, 'good good' men is equivalent to 'good' men." Only two symbols of number obey this formal law. They are 0 and 1. Their natural interpretation in the system of logic is *Nothing* and *Universe*, which are the two limits of class extension. If from the conception of the universe, as consisting of "men" and "not-men," we exclude or subtract the conception of "men," the resulting conception is that of the contrary class, "not-men." Hence, if x represent men, the class "not-men" will be represented by $1-x$. And, in general, whatever class of objects is represented by the symbol x, the contrary class will be expressed by $1-x$, which we may write $\bar{x}$. Boole now goes on to make the blunder referred to, in gravely stating: "Prop. IV. That axiom of metaphysicians which is termed the principle of contradiction, and which affirms that it is impossible for any being to possess a quality and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is $x^2=x$." As Mr. Venn has remarked, this "surely argues a strange inversion of order." Indeed, the inversion is so palpable that we are astonished to find Liard repeating the error on page 292 of his article, where he says, "*Maintenant il est aisé de voir que l'axiome appelé par les logicians* principe de contradiction, *et considéré par eux comme une loi primitive et irreductible de la pensée, est une conséquence de cette loi antérieure dont l'expression est:* $x^2=x$."

But while the law $x^2=x$ should have been introduced as rather the effect than the cause of the principle of contradiction, yet I believe I am announcing an important discovery when I say that it is this law alone which has, so far, rendered division impossible in the algebra of logic, which in turn forced Boole to introduce the machinery and all the features which have been objected to in his calculus. I may add, in passing, that, having traced the difficulty to its source, I believe myself able to overcome it, and hope to publish my solution at no distant day.

To return to our author, he says, p. 36: "Suppose it true that those members of a class, x, which possess a certain property, z, are identical with those members of a class, y, which possess the same property, z; it does not follow that the members of the class x universally are identical with the members of the class y. Hence it cannot be inferred from the equation $zx=zy$ that the equation $x=y$ is also true. In other words, the axiom of algebraists, that both sides of an equation may be divided by the same quantity, has no formal equivalent here." He attempts no explanation of this anomaly, but makes it analogous to the case where, in the algebra of number, if, in the equation $zx=zy$, z can be 0, we cannot deduce $x=y$. Now, this is an eminently false analogy, only representing the case where z is the limiting class "nought," which, combined with any class, gives nought. Here the two algebras are completely analogous, but this is not at all the point we are considering. The special limitation in logical algebra is not caused by any one special class, like 0, but applies to every class and to all equations, and has nothing in the slightest degree analogous to it in the algebra of number. When he reverts to this matter again, p. 88, we see more conclusively that he has been able to think of no logical cause for it, and can only fall back on this false quantitative analogy. He says:

"If the fraction $\frac{\acute{e}}{\acute{e}-e}$ has common factors in its numerator and denominator, we are not permitted to reject them, unless they are mere numerical constants. For the symbols x, y, etc., regarded as quantitative, may admit of such values, 0 and 1, as to cause the common factors to become equal to 0, in which case the algebraic rule of reduction fails. This is the case contemplated in our remarks on the failure of the algebraic axiom of division," p. 36. Now, if there was no cause for the failure of the division axiom except the reduction of some factor to nought, there would be no cause for calling attention to the matter, and we might proceed to use division precisely as we do when treating of number, since a zero has precisely the same effect in both algebras.

But, in point of fact, Boole cannot use real division at all. If he chooses to write $ayz=xz$ in the form $a=\frac{xz}{yz}$, he has not divided out any factor, and dare not. Even when he is certain z is not nought he cannot divide it out, which demonstrates instantly the falsity of his analogy. The real cause is the existence of the law, $xx=x^2=x$, in the logical algebra, which has no counterpart in that of number. That this is the true explanation appears very simply, as follows: If we have an equation in which a common factor appears in both members, as, *e. g.*, $zy=zx$, this law renders it impossible for us to know how far the class z coincides with x, since it may run from absolute difference up to complete identity; so that, in dividing out z, we may always be leaving some or all of it behind in the remaining factor. For example, if all rational white men = all white rational animals, and we divide out "rational," we have, all white men = all white animals. Now, the fact that this is not true, that a white man is not a white horse, though both are white animals, does not depend upon anything becoming zero, but upon the fact that on one side some of the meaning of rational has been unavoidably left behind in the term "men," while the division succeeded in taking it all out of the other member of the equation. If we start with the simple truth, "All men are all the rational animals," that is, $m=ra$, we may multiply both sides by r and it remains just as true; becoming $rm=rra=r^2a=ra \therefore rm=ra$. But, if now we attempt to divide out this r we just put in, it draws with it the original r from one member, while leaving it latent in the other member, and we have $m=a$, all men are all the animals.

This shows us why in Boole's system we cannot divide; and when, remembering this restriction, we use the fractional form, we get expressions which often bear on their face no meaning or interpretation. These Boole transforms, by what he calls development, into forms always strictly interpretable. The fact of his conducting his reasoning thus, through mediate uninterpretable steps, has been the most serious objection to his system, yet he saw no other way to attain a perfectly general solution.

This development theorem, given on p. 73, Prop. II, "To expand or develop a function involving any number of logical symbols," contains, and has been made, the basis of Stanley Jevons' whole logical system. Utterly misconceiving his master's attempt to give a genuine algebra of logic, which should make it a progressive science like quantitative mathematics, Mr. Jevons has been entirely content with the general method of indirect inference by trials, which is given immediately by this one theorem of development. We cannot enter here into a discussion of the principles involved in this process of generalized dichotomy. Merely as a hint at its application, we treat the simple proposition we have been using, $m=ra$. To get at what this can tell us about animals we express a as a function of m and r: $a=\frac{m}{r}$. Developing, we have $a=\frac{m}{r}=f\ (m.r)=f\ (1,1)\ m.r+f\ (1,0)\ m.\bar{r}+f\ (0,1)\ \bar{m}r+f\ (0,0)\ \bar{m}\bar{r}$. From this, without trials, Boole proves that all animals consist of all men and some irrational things not men. But, if he would have consented to use trials in referring to the premises in every particular instance, he would not have needed the co-efficients of his expansion. Thus, since all men are rational, the second term, $m.\bar{r}$, strikes out; and, since men are all the rational animals, the third term, $\bar{m}.r$, strikes out, and we are left for animals only mr and $\bar{m}\bar{r}$, as before. This satisfies Jevons. This he has adopted, and, as one instantly sees, it may be carried on without saying anything about expansion, and without putting the development in the form of an equation. For convenience, we may always use the same letters, taking as many as we need in regular alphabetical order, and denoting positive terms by capitals, and their negatives by small letters. In our example, let A = animal, B = man, C = rational, and, instead of developing only with reference to two terms, expand with reference to the three, and our constituents are eight in number, as follows:

1, ... ABC.
2, ... ABc.
3, ... AbC.
4, ... Abc.
5, ... aBC.
6, ... aBc.
7, ... abC.
8, ... abc.

Making our trial references to our premise, "all men are all the rational animals," 2, 3, 5, 6 strike out, and, selecting the terms left containing A, we have for animals only animals, men rational, and animals, not-men, not-rational, the same as before. If using, with Boole, the principle of quantification of the predicate, we express our premises in these same letters, the making of trial-references becomes purely mechanical, and thus Boole's theorem gave rise to Jevons' interesting logical machine. This, as a result, by the way, is certainly very charming, but the end and aim, a genuine satisfactory algebra of logic, should be kept steadily in view. It is overlooking this that makes even such an acute critic as Mr. Venn blame Boole for giving to the last process we shall mention, the process of getting rid of any terms we choose from our equations, the name "Elimination." Says Mr. Venn: "In each case no doubt a term disappears from the result, but the meaning and consequences of its disappearance are altogether distinct." Of course they are, but this is matter of *interpretation*, and to name the formal processes of a symbolic algebra according to interpretation would be in the highest degree unwise.

Here again our law of duality or simplicity, $x^2=x$, comes to our aid and makes the problem of elimination resolvable under all circumstances alike. In common algebra there exists a definite connection between the number of independent equations given and the number of symbols of quantity which it is possible to eliminate from them ; but, in the algebra of logic, from even a single equation an indefinite number of terms may be eliminated.

Here we will pause. We are now in position to see how it is that Boole's Logical Method can give an absolutely general solution to the final problem of practical logic. Its mode of application to every possible case is evident from the analogy of common algebra, and we may refer to the book itself for examples, instead of taking any here of sufficient intricacy to give any adequate idea of its astonishing grasp and power.

We have made no attempt at a complete presentation of the system. Our desire has been to call attention to the principles which rendered it possible, to show where its imperfection lies, to throw light on those points where his readers have been most apt to go astray, and to heighten the interest beginning to be widely shown in a truly wonderful work.